GENERALIZED BALANCED TOURNAMENT DESIGNS

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ABSTRACT. A generalized balanced tournament design, $GBTD(n, k)$, de-fined on a kn-set V, is an arrangement of the blocks of a $(kn, k, k - 1)$-$BIBD$ defined on V into an $n \times (kn - 1)$ array such that (1) every element of V is contained in precisely one cell of each column, and (2) every element of V is contained in at most k cells of each row. In this paper, we introduce $GBTD(n, k)s$ and describe connections between these designs and several other types of combinatorial designs. We also show how to use $GBTDs$ to construct resolvable, near resolvable, doubly resolvable and doubly near resolvable $BIBDs$.

1. INTRODUCTION

A generalized balanced tournament design, $GBTD(n, k)$, defined on a kn-set V, is an arrangement of the blocks of a $(kn, k, k-1)$-$BIBD$ defined on V into an $n \times (kn - 1)$ array such that

(1) every element of V is contained in precisely one cell of each column, and

(2) every element of V is contained in at most k cells of each row.

This generalizes the idea of a balanced tournament design and we note that a $GBTD(n, 2)$ is a $BTD(n)$, [32].

A simple but very important observation is stated in the next lemma.

Lemma 1.1. *Every element of a* $GBTD(n, k)$ *is contained* k *times in* $(n - 1)$ *rows and* $(k - 1)$ *times in the remaining row.*

Let G be a $GBTD(n, k)$. An element which is contained in only $k - 1$ cells of row i of G is called a deficient element of row i. It is easily seen that each row of G contains exactly k deficient elements. These elements are called the deficient k-tuple of row i. The deficient elements of row i need not occur in a common block of row i.

Lemma 1.2. *The deficient* k-tuples of G *partition the set* V *into* n k-tuples.

We illustrate these definitions by displaying a $GBTD(3, 3)$ in Figure 1. The

deficient triples of rows 1, 2 and 3 are respectively $\{4, 6, 8\}$, $\{1, 2, 9\}$ and $\{3, 5, 7\}$.

Let $C = (C_1, C_2, \ldots, C_n)^T$ where C_i, $1 \leq i \leq n$, is the deficient k-tuple of row i of G. If C occurs as a column in G $k - 1$ times, G is said to have property C. The $GBTD(3, 3)$ displayed in Figure 1 does not have property C. A $GBTD(4, 3)$ with Property C is displayed in Figure 2.

1 2 9	3 4 9	5 6 9	1 4 5	3 5 7	1 7 8	2 3 8	2 6 7
3 5 7	1 6 7	1 3 8	2 3 6	4 6 8	2 4 5	7 4 9	5 8 9
4 6 8	2 5 8	2 4 7	7 8 9	1 2 9	3 6 9	1 6 5	1 3 4

FIGURE 1. A $GBTD(3, 3)$

FGJ	EIL	DHK	FHL	EGK	DIJ	FIK	DGL	EHJ	ABC	ABC
AIL	CHK	CIJ	BGJ	BHL	AGK	AHJ	BIK	CGL	DEF	DEF
BEK	BFJ	AEL	CDK	ADJ	CFL	BDL	CEF	AFK	GHI	GHI
CDH	ADG	BFG	AEI	CFI	BEH	CEG	AFH	BDI	JKL	JKL

FIGURE 2. A $GBTD(4, 3)$ with Property C

A near generalized balanced tournament design, $NGBTD(n, k)$, defined on a $(kn+1)$-set V, is an arrangement of the blocks of a $(kn+1, k, k-1)$-$BIBD$ defined on V into an $n \times (kn + 1)$ array so that

(1) every element of V occurs precisely k times in each row;
(2) every column of the array contains kn distinct elements of V, and
(3) the columns form a near resolution of the $(kn + 1, k, k - 1)$-$BIBD$.

A near generalized balanced tournament design is a generalization of an odd balanced tournament design; in particular, a $NGBTD(n, 2)$ is an $OBTD(n)$, [19]. We illustrate this definition by displaying a $NGBTD(2, 3)$ in Figure 3.

124	235	346	450	561	602	013
356	460	501	612	023	134	245

FIGURE 3. A $NGBTD(2, 3)$

The existence of $BTD(n)s$ or $GBTD(n, 2)s$ was established in 1977 [32]; there exists a $GBTD(n, 2)$ for n a positive integer, $n \neq 2$. The existence of $OBTD(n)s$ or $NGBTD(n, 2)s$ is also well known and easy to prove [21]; there exists a $NGBTD(n, 2)$ for n a positive integer. Factored $BTDs$, partitioned $BTDs$, $BTDs$ with almost orthogonal resolutions and $OBTDs$ with orthogonal resolutions have been investigated extensively in the past few years. Results on these designs can be found in the following papers [15, 16, 17, 18, 8, 20,

21, 22]. Most of these designs have become of interest recently because of their relationships to other types of combinatorial designs. Of particular interest to us are the connections between balanced tournament designs and resolvable and doubly resolvable $BIBDs$ [19].

The purpose of this paper is to introduce generalized balanced tournament designs including factored and partitioned $GBTDs$, as well as near generalized balanced tournament designs. We will show that these designs are also of considerable interest because of their relationships to other designs. One of our main results will be to show that $PGBTD(n, k)s$ can be used directly to construct $DR(nk, k, k-1)$-$BIBDs$. This will provide a surprisingly easy way to construct the doubly resolvable designs. This paper is the first of a series of papers on generalized balanced tournament designs [10, 11]. The remaining papers describe in detail constructions for $GBTDs$ and existence results for these designs. As we noted above and in [20], existence results for $GBTD(n, k)s$ with $k = 2$ are known. In [11], we consider the next case, $k = 3$. For completeness, we list two of the existence results here.

Theorem 1.3 [11]. *For n a positive integer, $n \geq 3$, there exists a $GBTD(n, 3)$ except possibly for $n \in \{14, 17, 23, 47, 62, 68, 95, 102, 115\}$.*

Theorem 1.4 [11]. *For n a positive integer, there exists a $NGBTD(n, 3)$ except possibly for $n \in \{3, 38, 39, 118\}$.*

In the next section, we collect some preliminary definitions and results on resolvable, near resolvable, doubly resolvable and doubly near resolvable $BIBDs$. In §3, we define and discuss factored $GBTDs$ $(FGBTDs)$; we will show that $FGBTDs$ with Property C can be used to construct 1-dimensional frames. Partitioned $GBTDs$ $(PGBTDs)$ are defined in §4 and the equivalence of these designs to doubly resolvable designs is described. In §5, we generalize several constructions from [19] which use $GBTDs$ and $NGBTDs$ to construct resolvable, near resolvable, doubly resolvable and doubly near resolvable $BIBDs$.

2. PRELIMINARY DEFINITIONS AND RESULTS

Generalized balanced tournament designs can be used to construct resolvable, near resolvable, doubly resolvable and doubly near resolvable balanced incomplete block designs. In this section, we collect some of the necessary definitions and existence results for these constructions.

A balanced incomplete block design ($BIBD$) D is a collection B of subsets (blocks) taken from a finite v-set V of elements with the following properties:

    (1) Every pair of distinct elements from V is contained in precisely λ blocks of B.

    (2) Every block contains exactly k elements.

We denote such a design as a (v, k, λ)-$BIBD$. It is well known that necessary conditions for the existence of a (v, k, λ)-$BIBD$ are

$$(1) \qquad \lambda(v - 1) \equiv 0 \pmod{(k - 1)} \quad \text{and} \quad \lambda v(v - 1) \equiv 0 \pmod{k(k - 1)}.$$

A (v, k, λ)-*BIBD* D is said to be resolvable (and denoted by (v, k, λ)-*RBIBD*) if the blocks of D can be partitioned into classes $R_1, R_2, \ldots, R_r$ (resolution classes) where $r = \frac{\lambda(v-1)}{(k-1)}$ such that each element of D is contained in precisely one block of each class. The classes $R_1, R_2, \ldots, R_r$ form a resolution of D. Necessary conditions for the existence of a (v, k, λ)-*RBIBD* are that (1) be satisfied and that $v \equiv 0 \pmod{k}$.

A (v, k, λ)-*BIBD* D is said to be near resolvable (and denoted by $NR(v, k, \lambda)$-*BIBD*) if the blocks of D can be partitioned into classes $R_1, R_2, \ldots, R_v$ (resolution classes) such that for each element x of D there is precisely one class which does not contain x in any of its blocks and each class contains precisely $v - 1$ distinct elements of the design. Necessary conditions for the existence of $NR(v, k, \lambda)$-*BIBDs* are $v \equiv 1 \pmod{k}$ and $\lambda = k - 1$.

In general, the spectrum of resolvable and near resolvable (v, k, λ)-*BIBDs* remains open. The existence of these designs has been established for several small values of k and λ. In this paper, we are primarily interested in the cases $\lambda = k - 1$. For completeness, we include some of the results for other values of λ. It is well known that there exists a $(v, 2, 1)$-*RBIBD* for v a positive integer and $v \equiv 0 \pmod{2}$ and that there exists a $NR(v, 2, 1)$-*BIBD* for v a positive integer and $v \equiv 1 \pmod{2}$. The existence of *RBIBDs* and *NRBIBDs* has also been established for block size $k = 3$; the necessary conditions are also sufficient (with one exception).

Theorem 2.1 (Ray-Chaudhuri and Wilson [28]). *There exists a* $(v, 3, 1)$-*RBIBD if and only if* $v \equiv 3 \pmod{6}$.

Theorem 2.2 (Hanani [6]). (i) *There exists a* $(v, 3, 2)$-*RBIBD if and only if* $v \equiv 0 \pmod{3}$ *and* $v \neq 6$. (ii) *There exists a* $NR(v, 3, 2)$-*BIBD if and only if* $v \equiv 1 \pmod{3}$, $v \geq 4$.

For block size $k = 4$, the following results are known.

Theorem 2.3 (Hanani, Ray-Chaudhuri and Wilson [7]). *There exists a* $(v, 4, 1)$-*RBIBD if and only if* $v \equiv 4 \pmod{12}$.

Theorem 2.4 (Baker [1]). *There exists a* $(v, 4, 3)$-*RBIBD if and only if* $v \equiv 0 \pmod{4}$.

For block size larger than 4, very little is known. Recent results for $(v, 5, 1)$-*RBIBDs* can be found in [2], and results for $(v, 6, 10)$-*RBIBDs* appear in [1]. For the designs that we are interested in with $\lambda = k - 1$, the next case where some results have been obtained is $k = 8$.

Theorem 2.5 (Rokowska [31]). *For all* $v \equiv 0 \pmod{8}$ *which are not divisible by* 3, 5 *and* 7, *there exists a* $(v, 8, 7)$-*RBIBD*.

Some other infinite classes of resolvable *BIBDs* are known to exist. We include some existence results for the case $\lambda = k - 1$ in our list of results for doubly resolvable *BIBDs*.

A (v, k, λ)-*BIBD* is said to be doubly (near) resolvable if there exist two (near) resolutions R and R' of the blocks such that $|R_i \cap R'_j| \leq 1$ for all $R_i \in R$, $R'_j \in R'$. (It should be noted that the blocks of the design are considered as being labelled so that if a subset of the elements occurs as a block more then once the blocks are treated as distinct.) The (near) resolutions R and R' are called orthogonal resolutions of the design. A doubly resolvable (v, k, λ)-*BIBD* is denoted by $DR(v, k, \lambda)$-*BIBD* and a doubly near resolvable (v, k, λ)-*BIBD* by $DNR(v, k, \lambda)$-*BIBD*.

A Kirkman square with block size k, order v and index λ, $KS_k(v; 1, \lambda)$, is a $t \times t$ array K ($t = \lambda(v-1)/(k-1)$) defined on a v-set V such that

(1) each element of V is contained in precisely one cell of each row and column of K;

(2) each cell of K is either empty or contains a k subset of V;

(3) The collection of blocks obtained from the nonempty cells of K is a (v, k, λ)-*BIBD*.

We can use a pair of orthogonal resolutions of a $DR(v, k, \lambda)$-*BIBD* to construct a $KS_k(v; 1, \lambda)$. The rows of the array form one resolution of the $DR(v, k, \lambda)$-*BIBD* and the columns form an orthogonal resolution. Similarly, we can use a pair of orthogonal resolutions of a $DNR(v, k, \lambda)$-*BIBD* to construct a $v \times v$ array. The rows of the array will form one resolution of the design and the columns will form an orthogonal resolution. If the $DNR(v, k, \lambda)$-*BIBD* has the additional property that under an appropriate ordering of the resolution classes R and R', $R_i \cup R'_i$ contains precisely $v - 1$ distinct elements of the design for all i, then the array is called a $(1, \lambda; k, v, 1)$-frame [24]. Note that the diagonal of a $(1, \lambda; k, v, 1)$-frame is empty and a unique element of the design can be associated with each cell (i, i). (This distinction between $(1, \lambda; k, v, 1)$-frames and $DNR(v, k, \lambda)$-*BIBDs* is important in recursive constructions.)

Even less is known about the spectrum of doubly resolvable and doubly near resolvable (v, k, λ)-*BIBDs*. Although several infinite classes of $DR(v, k, \lambda)$-*BIBDs* are known for $k \geq 3$, [5, 3, 38], the existence of $DR(v, k, \lambda)$-*BIBDs* has been settled only for $k = 2$ and $\lambda = 1$. ($DR(v, 2, 1)$-*BIBDs* are also called Room squares.)

Theorem 2.6 (Mullin and Wallis [27]). *There exists a* $DR(v, 2, 1)$-*BIBD for* v *a positive integer,* $v \equiv 0 \pmod{2}$ *and* $v \neq 4$ *or* 6.

An immediate corollary of this result is the following.

Corollary 2.7. *There exists a* $DNR(v, 2, 1)$-*BIBD* (*and a* $(1, 1; 2, v, 1)$-*frame*) *for* $v \equiv 1 \pmod{2}$, $v \neq 3$ *or* 5.

For block size $k = 3$ the problem is more difficult. The best result, thus far, for $\lambda = 1$ is asymptotic.

Theorem 2.8 (Rosa and Vanstone [30]). *There exists a constant* v_1 *such that for all* $v_1 \leq v$ *and* $v \equiv 3 \pmod{6}$ *there exists a* $DR(v, 3, 1)$-*BIBD*.

The problem is more tractable for $k = 3$ and $\lambda = 2$. The best results, thus far, for $DNR(v, 3, 2)$-$BIBDs$ are the following.

Theorem 2.9 (Lamken and Vanstone [24]). *For n a positive integer, there exists a $(1, 2; 3, 15n + 1, 1)$-frame or a $DNR(15n + 1, 3, 2)$-$BIBD$.*

Theorem 2.10 (Lamken [13]). *There exists a $DNR(v, 3, 2)$-$BIBD$ for $v \equiv 1 \pmod 9$ except possibly for $v = 55$.*

There are two analogous results for $DR(v, 3, 2)$-$BIBDs$.

Theorem 2.11 (Lamken and Vanstone [23]). *For $v \equiv 3 \pmod{12}$, there exists a $DR(v, 3, 2)$-$BIBD$ which contains as a subarray a $DR(3, 3, 2)$-$BIBD$.*

Theorem 2.12 (Lamken [13]). *Let $M = \{6, 8, 10, 12, 14, 15, 18, 20, 22, 24, 26, 30, 32, 34, 38, 39, 42, 44, 46, 48, 51, 52, 58, 118\}$, and let $v = 9m + 3$ where m is a positive integer, $m \notin M$. Then there exists a $DR(v, 3, 2)$-$BIBD$.*

Quite recently, the spectrum of $DR(v, 3, 2)$-$BIBDs$ has been determined with a small number of possible exceptions, [12]. We note some of the ideas used for this result in §4.

For $k \geq 3$, some infinite classes of $DR(v, k, \lambda)$-$BIBDs$ have been constructed. We include two of these results.

Theorem 2.13 (Fuji-Hara and Vanstone [5]). *For p a prime power and n a positive integer greater than 2, there exists a $DR(p^n, p, 1)$-$BIBD$.*

Theorem 2.14 (Curran and Vanstone [3]). *For k a prime power and $k \geq 3$ there exists a $DR(k^2 + k, k, k - 1)$-$BIBD$.*

Finally, we note that the generalization of the Kirkman square has been studied in [14, 25, 26]. Frames have also been investigated in a more general setting; we refer to [4, 24, 37, 38, 36] for some of these results.

3. FACTORED GENERALIZED BALANCED TOURNAMENT DESIGNS

Let G be a $GBTD(n, k)$ defined on V, and let $C = (C_1, C_2, \ldots, C_n)^T$ where C_i, $1 \leq i \leq n$, be the deficient k-tuple of row i of G. Suppose the blocks in row i of $G \cup C_i$ can be partitioned into k sets of n blocks each, $F_{i1}, F_{i2}, \ldots, F_{ik}$, so that every element of V occurs precisely once in F_{ij} for $j = 1, 2, \ldots, k$. If every row of G has this property, then G is called a factored generalized balanced tournament design and is denoted by $FGBTD(n, k)$. We will call $F_{i1}, F_{i2}, \ldots, F_{i(k-1)}, F_{ik}$ the factors of row i. The $GBTD(4, 3)$ displayed in Figure 2 is factored; the factors are listed in Table 1. We note that if a $FGBTD(n, k)$ also has Property C then $C_i \in F_{ij}$ for $j = 1, 2, \ldots, k$ and $i = 1, 2, \ldots, n$.

The existence of factored generalized balanced tournament designs with Property C is equivalent to the existence of 1-dimensional frames with or-

thogonal resolutions. We will need some additional definitions and notation in order to describe these frames.

<div align="center">

TABLE 1

Factors for the $FGBTD(4, 3)$ displayed in Figure 2

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Row 1:	ABC	FGJ	EIL	DHK
	ABC	FHL	EGK	DIJ
	ABC	FIK	DGL	EHJ
Row 2:	DEF	BIK	AHJ	CGL
	DEF	AIL	CHK	BGJ
	DEF	CIJ	BHL	AGK
Row 3:	GHI	BEK	ADJ	CFL
	GHI	BFJ	AEL	CDK
	GHI	BDL	CEJ	AFK
Row 4:	JKL	CDH	BFG	AEI
	JKL	AFH	CEG	BDI
	JKL	BEH	CFI	ADG

A group divisible design (GDD) is a collection B of subsets (blocks) of size k taken from a v-set V along with a partition of V in groups $G_1, G_2, \ldots, G_n$ such that

(1) any two elements from distinct groups are contained in precisely λ_2 blocks of B; and

(2) any two distinct elements from the same groups are contained in precisely λ_1 blocks of B $(\lambda_1 < \lambda_2)$.

We denote such a design by $GDD(v; k; G_1, G_2, \ldots, G_n; \lambda_1, \lambda_2)$. If $|G_i| = h$ for $i = 1, 2, \ldots, n$, we denote the design by $GDD(v; k; h; \lambda_1, \lambda_2)$.

A GDD is resolvable if the blocks can be partitioned into classes (resolution classes) $R_1, R_2, \ldots, R_r$ such that every element of V is contained in precisely one block of each R_i, $1 \leq i \leq r$. The collection of resolution classes is called a resolution of the GDD.

Lemma 3.1 (Vanstone [38]). *In a resolvable GDD all groups have the same size.*

We denote a resolvable GDD with $|G_i| = h$ for all i by $RGDD(v; k; h; \lambda_1, \lambda_2)$. Let V be a set of nk elements. Let $G_1, G_2, \ldots, G_n$ be a partition of V into n sets where $|G_i| = k$ for $i = 1, 2, \ldots, n$. A 1-dimensional $\{G_1, G_2, \ldots, G_n\}$-frame F with block size k, index $k - 1$ and group size k is a set P of partial resolution classes of V which satisfies the following properties. Let $P = \{P_{i1}, P_{i2}, \ldots, P_{ik} \,|\, i = 1, 2, \ldots, n\}$.

(1) Each partial resolution class P_{ij} contains $n - 1$ blocks of size k.

(2) Every element of $V - G_i$ occurs precisely once in P_{ij} for $j = 1, 2$, $\ldots, n$.

(3) The collection of blocks in P is a $GDD(nk; k; G_1, \ldots, G_n; 0, k-1)$.

We will call F a 1-dimensional $(1, k-1; k, n, k)$-frame. (This notation is consistent with our notation for frames in general [24].) A 1-dimensional $(1, 2; 3, 4, 3)$-frame is displayed in Figure 4 to illustrate this definition. We note that 1-dimensional frames have become of interest in the past few years and have been used to prove several new results for resolvable designs [29, 35, 36].

Let G be a $RGDD(nk; k; G_1, G_2, \ldots, G_n; 0, k-1)$ defined on a set V. Let $R_1, R_2, \ldots, R_{(n-1)k-2}$ denote the resolution classes of a resolution R of G. Suppose there exists a 1-dimensional $(1, k-1; k, n, k)$-frame F where F is a $\{G_1, G_2, \ldots, G_n\}$-frame and F contains the blocks of G. Let $P = \{P_{i1}, P_{i2}, \ldots, P_{ik} | i = 1, 2, \ldots, n\}$ denote the partial resolution classes of F. If $|R_i \cap (P_{j1} \cup P_{j2} \cup \cdots \cup P_{jk})| = 1$ for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, nk - k - 2$, then we say R is an orthogonal resolution for F. As an example of this definition, the first nine columns of the $FGBTD(4, 3)$ in Figure 2 form an orthogonal resolution for the 1-dimensional frame in Figure 4.

P_{11}	FGJ	EIL	DHK
P_{12}	FHL	EGK	DIJ
P_{13}	FIK	DGL	EHJ
P_{21}	BIK	AHJ	CGL
P_{22}	AIL	CHK	BGJ
P_{23}	CIJ	BHL	AGK
P_{31}	BEK	ADJ	CFL
P_{32}	BFJ	AEL	CDK
P_{33}	BDL	CEJ	AFK
P_{41}	CDH	BFG	AEI
P_{42}	AFH	CEG	BDI
P_{43}	BEH	CFI	ADG

FIGURE 4. A 1-dimensional $(1, 2; 3, 4, 3)$-frame

We can now prove the following equivalence.

Theorem 3.2. *There exists a $FGBTD(n, k)$ with Property C if and only if there exists a 1-dimensional $(1, k-1; k, n, k)$-frame F with an orthogonal resolution.*

Proof. Let G be a $FGBTD(n, k)$ with Property C defined on a set V of nk elements. Suppose the factors of row i are $F_{i1}, F_{i2}, \ldots, F_{i(k-1)}, F_{ik}$ for $i = 1, 2, \ldots, n$. Since G has Property C, $C_i \in F_{ij}$ for $j = 1, 2, \ldots, k-1$.

Let $F'_{ij} = F_{ij} - C_i$ for $j = 1, 2, \ldots, k - 1$ and let $F'_{ik} = F_{ik}$. The collection of blocks in $\bigcup_{i=1,2,\ldots,n; j=1,2,\ldots k} F'_{ij}$ is a $GDD(nk; k; k; 0, k - 1)$. The groups of the design are the deficient k-tuples, $C_1, C_2, \ldots, C_n$. Every element in $V - C_i$ occurs precisely once in the blocks of F'_{ij}, $j = 1, 2, \ldots, k$. We can construct a 1-dimensional $(1, k - 1; k, n, k)$-frame F by defining the partial resolution classes as follows: $P_{ij} = F'_{ij}$ for $j = 1, 2, \ldots, k$ and $i = 1, 2, \ldots, n$. If we delete the $k - 1$ copies of C from G, it is clear that the remaining columns of G form resolution classes for a resolution R of the resulting $GDD(nk; k; k; 0, k - 1)$. Since each resolution class contains precisely one block from each row of the $FGBTD(n, k)$ and thus precisely one block from $(P_{j1} \cup P_{j2} \cup \cdots \cup P_{jk})$, the resolution R is an orthogonal resolution for the 1-dimensional frame F.

This construction can be reversed. If we start with a 1-dimensional $(1, k-1; k, n, k)$-frame with an orthogonal resolution, we can construct a $FGBTD(n, k)$ as follows. Let F be a $\{C_1, C_2, \ldots, C_n\}$-frame defined on an nk set V. The blocks in row i of the $FGBTD(n, k)$ will be the blocks in $P_{i1} \cup P_{i2} \cup \cdots \cup P_{ik}$ together with $k - 1$ copies of the group C_i. The ith resolution class of the orthogonal resolution R, R_i, will contain the blocks of the ith column of the $FGBTD(n, k)$ for $i = 1, 2, \ldots, nk - k - 2$. Finally, the $FGBTD(n, k)$ will contain $k - 1$ copies of the column $C = (C_1, C_2, \ldots, C_n)^T$. □

Corollary 3.3. *If there exists a $FGBTD(n, k)$ with Property C, then there exists a $(1, k - 1; k, n, k)$-frame.*

The spectrum of $FGBTD(n, k)$ with Property C has been determined for $k = 2$ and for $k = 3$ with a finite number of possible exceptions.

Theorem 3.3 [15]. *For n a positive integer, $n \neq 2$ or 4, there exists a $FGBTD$ $(n, 2)$ with Property C.*

Theorem 3.4 [11]. *For n a positive integer, $n \geq 4$, there exists a $FGBTD(n, 3)$ with Property C except possibly for $n \in \{14, 15, 17, 18, 19, 23, 27, 31, 39, 47, 62, 68, 87, 102, 103, 114, 115, 127, 131, 151, 159\}$.*

4. Partitioned generalized balanced tournament designs

Let G be a $GBTD(n, k)$ defined on V. Suppose we can partition the columns of G into $k + 1$ sets $B_1, B_2, \ldots, B_{k+1}$ where $|B_i| = n$ for $i = 1, 2, \ldots, k - 2$, $|B_i| = n - 1$ for $i = k - 1, k$ and $|B_{k+1}| = 1$ such that

   (1) every element of V occurs precisely once in each row and column of
       B_i for $i = 1, 2, \ldots, k - 2$, and
   (2) every element of V occurs precisely once in each row and column of
       $B_i \cup B_{k+1}$ for $i = k - 1$ and $i = k$.

Then the $GBTD(n, k)$, G, is called partitioned and we denote the design by $PGBTD(n, k)$. Let $C = (C_1, C_2, \ldots, C_n)^T$ where C_i, $1 \leq i \leq n$, is the deficient k-tuple of row i of G. It is clear that if G is partitioned then

0 1 3	9 13 18	14 20 9	β 16 8	α 17 2	7 10 11	γ 6 15	5 12 19
8 14 19	β 10 2	α 11 17	1 4 5	γ 0 9	15 16 18	3 7 12	6 13 20
α 5 11	16 19 20	γ 15 3	9 10 12	18 1 6	3 8 13	β 4 17	7 14 0
γ 9 18	3 4 6	12 16 0	17 2 7	β 19 11	α 20 5	10 13 14	8 15 1
6 10 15	11 17 1	β 13 5	α 14 20	4 7 8	γ 3 12	18 19 0	9 16 2
β 7 20	α 8 14	19 1 2	γ 18 6	12 13 15	0 4 9	6 11 16	10 17 3
13 16 17	γ 12 0	6 7 9	15 19 3	20 5 10	β 1 14	α 2 8	11 18 4
2 4 12	5 7 15	8 10 18	11 13 0	14 16 3	17 19 6	20 1 9	α β γ

B_1

14 17 18	γ 13 1	7 8 10	16 20 4	0 6 11	β 2 15	α 3 9
1 2 4	10 14 19	15 0 5	β 17 9	α 18 3	8 11 12	γ 7 16
9 15 20	β 11 3	α 12 18	2 5 6	γ 1 10	16 17 19	4 8 13
α 6 12	17 20 0	γ 16 4	10 11 13	19 2 7	3 9 14	β 5 18
γ 10 19	4 5 7	13 17 1	18 3 8	β 20 12	α 0 6	11 14 15
7 11 16	12 18 2	β 14 6	α 15 0	5 8 9	γ 4 13	19 20 1
β 8 0	α 9 15	20 2 3	γ 19 7	13 14 16	1 5 10	6 12 17
3 5 13	6 8 16	9 11 19	12 14 1	15 17 4	18 20 7	0 2 10

B_2

β 9 1	α 10 16	0 3 4	γ 20 8	14 15 17	2 6 11	7 13 18	5 12 19
15 18 19	γ 14 2	8 9 11	17 0 5	1 7 12	β 3 16	2 4 10	6 13 20
2 3 5	11 15 20	16 1 6	β 18 10	α 19 4	9 12 13	γ 8 17	7 14 0
10 16 0	β 12 4	α 13 19	3 6 7	γ 2 11	17 18 20	5 9 14	8 15 1
α 7 13	18 0 1	γ 17 5	11 12 14	20 3 8	4 10 15	β 6 19	9 16 2
γ 11 20	5 6 8	14 18 2	19 4 9	β 0 13	α 1 7	12 15 16	10 17 3
8 12 17	13 19 3	β 15 7	α 16 1	6 9 10	γ 5 14	20 0 2	11 18 4
4 6 14	7 9 17	10 12 20	13 15 2	16 18 5	19 0 8	1 3 11	α β γ

B_3 B_4

FIGURE 5. $P = [B_1 \ B_2 \ B_3 \ B_4]$

$B_{k+1} = C$. If G is partitioned and has Property C, then each of the $n \times n$ arrays $B_1, B_2, \ldots, B_{k-2}$ will contain a copy of C as a column. We illustrate this definition by displaying a $PGBTD(8, 3)$ in Figure 5. This design also has Property C.

The existence of $PGBTDs$ is equivalent to the existence of Kirkman squares in diagonal form. Let K be a $KS_k(nk; 1, k-1)$ defined on V. K is said to be in diagonal form if it can be written in the following form.

$$K = \begin{bmatrix} A & F_1 & & & & & \\ F_2 & E & & & & & \\ & & D_1 & & & & \\ & & & D_2 & & & \\ & & & & \ddots & & \\ & & & & & D_{k-2} \end{bmatrix}$$

FIGURE 6. A $KS_3(24; 1, 2)$ (B_1, B_2, B_3 are displayed in Figure 5)

A is an $n \times n$ array which contains n k-tuples along its diagonal. F_1 is an $n \times n - 1$ array of k-tuples; F_2 is an $n - 1 \times n$ array of k-tuples; and E is an $n - 1 \times n - 1$ empty array. Each of the arrays D_i, $1 \le i \le k - 2$, is an $n \times n$ array of k-tuples. All of the remaining cells of K are empty. We recall that K displays a pair of orthogonal resolutions for a $DR(v, k, k - 1)$-$BIBD$.

Theorem 4.1. *There exists a $PGBTD(n, k)$ if and only if there exists a $KS_k(nk; 1, k - 1)$ in diagonal form.*

Proof. Let P be a $PGBTD(n, k)$ defined on V. Let $B_1, B_2, \dots, B_{k+1}$ be a corresponding partitioning of the columns of P. Let $C = (C_1, C_2, \dots, C_n)^T$ where C_i, $1 \le i \le n$, is the deficient k-tuple of row i of P. Recall that $B_{k+1} = C$. We now define the arrays $A, F_1, F_2, D_1, D_2, \dots, D_{k-2}$ which provide the diagonal form of a $KS_k(nk; 1, k - 1)$, K, defined on V. A is an $n \times n$ array which contains C_i in cell (i, i). (All of the other cells of

A are empty.) Let $F_1 = B_k$ and let $F_2 = B_{k-1}^T$. Finally, let $D_i = B_i$ for $i = 1, 2, \ldots, k-2$. It is easy to verify that we have constructed the arrays for a $KS_k(nk; 1, k-1)$ in diagonal form.

The construction can be reversed. Let K be a $KS_k(nk; 1, k-1)$ defined on V. Let $C = B_{k+1} = (C_1, C_2, \ldots, C_n)^T$ where C_i is the k-tuple contained in cell (i, i) of A. It is straightforward to verify that $[D_1 D_2 \cdots D_{k-2} F_2^T F_1 C]$ is a $PGBTD(n, k)$ defined on V. $\square$

A $KS_3(24; 1, 2)$ in diagonal form which is equivalent to the $PGBTD(8, 3)$ in Figure 5 is displayed in Figure 6.

We note that for $k = 2$ this construction reduces to the construction used by D. R. Stinson in [34] to provide the connection between Room squares and $PBTDs$.

Corollary 4.2 (D. R. Stinson [34]). *There exists a $PGBTD(n, 2)$ if and only if there exists a $MESRS(2n - 1)$.*

The following result has been established for $k = 2$.

Theorem 4.3 (Lamken and Vanstone [16], [17], [18], Lamken [8]). *There exists a $PGBTD(n, 2)$ for all $n \geq 5$ except possibly for $n \in \{9, 11, 15, 26, 28, 34, 44\}$.*

Constructions and existence results for $PGBTD(n, 3)s$ have been used quite recently to determine the spectrum of $DR(3n, 3, 2)$-$BIBDs$ or $KS_3(3n; 1, 2)s$, [12]. There are now only a small number of possible exceptions for the existence of $KS_3(3n; 1, 2)s$; these results are discussed in detail in [12]. We refer to [10, 11, 12] for constructions and results for $PGBTD(n, k)s$ with $k \geq 3$.

5. Using $GBTDs$ to construct resolvable $BIBDs$

In this section, we generalize the constructions in [19] to use generalized balanced tournament designs to construct resolvable, near resolvable, doubly resolvable and doubly near resolvable $BIBDs$. The first three constructions are for resolvable and near resolvable designs.

Theorem 5.1. *If there exists a $GBTD(n, k)$, a $(kn+1, k+1, 1)$-$RBIBD$ and a $NR(n, k+1, k)$-$BIBD$, then there is a $((k+1)n, k+1, k)$-$RBIBD$.*

Proof. Let $V_1 = \{x_{i1}, x_{i2}, \ldots, x_{ik} | i = 1, 2, \ldots, n\}$ and let $V_2 = \{y_1, y_2, \ldots, y_n\}$.

Let G' be a $GBTD(n, k)$ defined on V_1. Suppose the deficient k-tuple of row i in G' is $\{x_{i1}, x_{i2}, \ldots, x_{ik}\}$ for $i = 1, 2, \ldots, n$. Let D' be a resolvable $(kn+1, k+1, 1)$-$BIBD$ defined on $V \cup \{\infty\}$ so that the blocks containing ∞ are $\{\infty, x_{i1}, x_{i2}, \ldots, x_{ik}\}$ for $i = 1, 2, \ldots, n$. Let D_i' be the resolution class of D which contains the block $\{\infty, x_{i1}, x_{i2}, \ldots, x_{ik}\}$ for $i = 1, 2, \ldots, n$. N will denote a NR $(n, k+1, k)$-$BIBD$ defined on V_2 and N_i will denote the resolution class of N which does not contain the element y_i.

We construct a resolvable $(kn + n, k + 1, k)$-$BIBD$ on $V_1 \cup V_2$ as follows. To each block in row i of G' add the element y_i $(i = 1, 2, \ldots, n)$. Denote the resulting array of blocks of size $k + 1$ by G. Let $C_1, C_2, \ldots, C_{nk-1}$ be the columns of G. Replace each block $\{\infty, x_{i1}, x_{i2}, \ldots, x_{ik}\}$ in D' with the block $\{y_i, x_{i1}, x_{i2}, \ldots, x_{ik}\}$ for $i = 1, 2, \ldots, n$. D will denote the resulting configuration. Let D_i be the corresponding resolution class of D which contains the block $\{y_i, x_{i1}, x_{i2}, \ldots, x_{ik}\}$ $(i = 1, 2, \ldots, n)$.

The blocks in $G \cup D \cup N$ form a $(kn + n, k + 1, k)$-$BIBD$. Every distinct pair in V_1 occurs $k - 1$ times in G and once in D. Every pair $\{y_i, x_{j\ell}\}$ occurs k times in $G \cup D$. Every pair in V_2 occurs k times in N. It is easy to verify that $\{C_1, C_2, \ldots, C_{nk-1}, D_1 \cup N_1, D_2 \cup N_2, \ldots, D_n \cup N_n\}$ is a resolution for this $(kn + n, k + 1, k)$-$BIBD$ defined on $V_1 \cup V_2$. $\square$

The following example illustrates this theorem. Since there exists a $GBTD(5, 3)$ [19], a $(16, 4, 1)$-$RBIBD$ [7] and a $NR(5, 4, 3)$-$BIBD$, we can construct a $(20, 4, 3)$-$RBIBD$.

Theorem 5.2. *If there exists a $GBTD(n, k)$, a $RGDD_{n-1}(nk; k + 1; k; 0, 1)$ and an $(n, k + 1, k)$-RBIBD, then there is a $(kn + n, k + 1, k)$-RBIBD.*

Proof. Let $V_1 = \{x_{i1}, x_{i2}, \ldots, x_{ik} | i = 1, 2, \ldots, n\}$ and let $V_2 = \{y_1, y_2, \ldots, y_n\}$.

Let G' be a $GBTD(n, k)$ defined on V_1. Suppose the deficient k-tuple of row i in G' is $\{x_{i1}, x_{i2}, \ldots, x_{ik}\}$ for $i = 1, 2, \ldots, n$. Let D be a $RGDD_{n-1}(nk; k + 1; k; 0, 1)$ defined on V_1 so that the groups of D are the k-tuples $\{x_{i1}, x_{i2}, \ldots, x_{ik}\}$ for $i = 1, 2, \ldots, n$. Let $D_1, D_2, \ldots, D_{n-1}$ be the resolution classes of D. Let N denote an $(n, k + 1, k)$-$RBIBD$. $N_1, N_2, \ldots, N_{n-1}$ will be the resolution classes of N.

We construct a $(kn + n, k + 1, k)$-$RBIBD$ as follows. To each block in row i of G' add the elements y_i $(i = 1, 2, \ldots, n)$. Denote the resulting array of blocks of size $k + 1$ by G. Let $C_1, C_2, \ldots, C_{nk-1}$ be the columns of G. Let $C_{nk} = \{\{y_i, x_{i1}, x_{i2}, \ldots, x_{ik}\} | i = 1, 2, \ldots, n\}$. The blocks in $G \cup D \cup N \cup C_{nk}$ form a $(kn + n, k + 1, k)$-$BIBD$. Every distinct pair in V_1 occurs $k - 1$ times in G and once in $D \cup C_{nk}$. Every pair $\{y_i, x_{j\ell}\}$ occurs k times in $G \cup C_{nk}$. Every pair in V_2 occurs k times in N. It is easy to verify that $\{C_1, C_2, \ldots, C_{nk}, D_1 \cup N_1, D_2 \cup N_2, \ldots, D_{n-1} \cup N_{n-1}\}$ is a resolution for this $(kn + k, k + 1, k)$-$BIBD$ defined on $V_1 \cup V_2$. $\square$

The following is an example of this construction. Since a $GBTD(8, 3)$ exists [11], a $RGDD_7(24; 4; 3; 0, 1)$ exists [34] and an $(8, 4, 3)$-$RBIBD$ is easily constructed, we can use Theorem 5.2 to construct a $(32, 4, 3)$-$RBIBD$.

Theorem 5.3. *If there is a $NGBTD(n, k)$, a $(kn + 1, k + 1, 1)$-RBIBD and a $NR(n, k + 1, k)$-BIBD, then there is a $NR(kn + n + 1, k + 1, k)$-BIBD.*

Proof. Let $V_1 = \{x_1, x_2, \ldots, x_{nk+1}\}$ and let $V_2 = \{y_1, y_2, \ldots, y_n\}$.

Let G' be a $NGBTD(n, k)$ defined on V_1. To each block in row i of G'

add the element y_i, $i = 1, 2, \ldots, n$. Denote the resulting array of blocks of size $k + 1$ by G and label the columns of G, $G_1, G_2, \ldots, G_{nk+1}$. Suppose the columns are arranged so that G_i contains every element of $V_1 \cup V_2 - \{x_i\}$ precisely once.

Let D be a $(kn+1, k+1, 1)$-RBIBD defined on V_1. Let $D_1, D_2, \ldots, D_n$ be a set of resolution classes for D. N will denote a $NR(n, k+1, k)$-BIBD defined on V_2 and N_i will denote the resolution class of N which does not contain y_i.

The blocks in $G \cup D \cup N$ form a $(kn + n + 1, k+1, k)$-BIBD. Each pair of distinct elements from V_1 occurs $k - 1$ times in G and once in D. Each pair of distinct elements from V_2 occurs k times in N. Finally, each pair of the form $\{x_i, y_j\}$, $1 \le i \le nk + 1$ and $1 \le j \le n$, occurs k times in G. A near resolution of this design is given by the following (near) resolution classes: $G_1, G_2, \ldots, G_{nk+1}, D_1 \cup N_1, D_2 \cup N_2, \ldots, D_n \cup N_n$. □

A simple example of this construction is as follows. There exists a $NGBTD(5, 3)$ [11], a $(16, 4, 1)$-RBIBD [7] and a $NR(5, 4, 3)$-BIBD. We can use Theorem 5.3 to construct a $NR(21, 4, 3)$-BIBD.

In [19] we used BTDs and OBTDs to construct doubly resolvable and doubly near resolvable BIBDs with block size 3. We can generalize these constructions to use GBTDs and NGBTDs with block size k to produce doubly resolvable and doubly near resolvable BIBDs with block size $k + 1$. In order to describe these constructions, we require several definitions.

Let G be a $NGBTD(n, k)$. Let $R_1, R_2, \ldots, R_n$ be the rows of G and let $G_1, G_2, \ldots G_{nk+1}$ be the columns of G. $G' = \{G_1, G_2, \ldots, G_{nk+1}\}$ is a (near) resolution of the underlying $(kn + 1, k, k - 1)$-BIBD. A resolution D, $D = \{D_1, D_2, \ldots, D_{kn}\}$, is called an orthogonal resolution to G' if

(i) $|G_i \cap D_j| \le 1$ for $1 \le i, j \le kn + 1$ and
(ii) $|D_j \cap R_i| = 1$ for $1 \le j \le kn + 1$, $1 \le i \le n$.

If D exists, we say that the $NGBTD(n, k)$ has a pair of orthogonal resolutions (ORs).

For GBTDs, as in the case of BTDs, we define a pair of almost orthogonal resolutions. Let G be a $GBTD(n, k)$. Let $R_1, R_2, \ldots, R_n$ be the rows of G and let $G_1, G_2, \ldots G_{kn-1}$ be the columns of G. $G' = \{G_1, G_2, \ldots, G_{kn-1}\}$ is a resolution of the underlying $(kn, k, k - 1)$-BIBD. A resolution $D = \{D_1, D_2, \ldots, D_{kn-1}\}$ will be called almost orthogonal to G' if

(i) $G_{kn-1} = D_{kn-1}$,
(ii) $|G_i \cap D_j| \le 1$ for $1 \le i, j \le kn - 2$,
(iii) $|D_j \cap R_i| = 1$ for $1 \le j \le kn - 2$, $1 \le i \le n$.

If D exists, we say that G has a pair of almost orthogonal resolutions (denoted by AORs). If G is a $GBTD(n, k)$ with a pair of almost orthogonal resolutions with the property that the deficient k-tuples of G are contained in the shared

resolution class G_{kn-1}, then we say that G has property C'.

Our constructions will also require the existence of $KS_{k+1}(kn + 1; 1, 1)s$ with complementary $(1, k; k + 1, n, 1)$-frames.

Let K be a $KS_{k+1}(kn + 1; 1, 1)$ defined on $V \cup \{\infty\}$ where ∞ occurs in each cell of the main diagonal.

We say K has a complementary $(1, k; k+1, n, 1)$-frame (or a complementary $DNR(n, k + 1, k)$-$BIBD$) if there exists a $(1, k; k + 1, n, 1)$-frame (or a $DNR(n, k + 1, k)$-$BIBD$) which can be written in the empty cells of K.

Theorem 5.4. *If there is a $GBTD(n, k)$ with a pair of almost orthogonal resolutions and Property C' and if there is a $KS_{k+1}(kn + 1; 1, 1)$ with a complementary $(1, k; k + 1, n, 1)$-frame, then there is a $KS_{k+1}(kn + n + 1; 1, k)$ or a $DR(kn + n + 1, k + 1, k)$-$BIBD$.*

Proof. Let $V_i = \{x_{i1}, x_{i2}, \ldots, x_{ik}\}$ for $i = 1, 2, \ldots, n$ and let $V = \bigcup_{i=1}^{n} V_i$. Let $W = \{y_1, y_2, \ldots, y_n\}$.

Let G' be a $GBTD(n, k)$ defined on V with a pair of almost orthogonal resolutions and Property C'. Suppose the deficient k-tuples of G' are V_i for $i = 1, 2, \ldots, n$. Let $C = (C_1, C_2, \ldots, C_n)^T$ be the column of deficient k-tuples where C_i is the deficient k-tuple for row i. Since G' has Property C', $V_i = C_j$ for some j. Denote the columns of G' by $A'_1, A'_2, \ldots, A'_{kn+1}$. Let $D = \{D'_1, \ldots, D'_{kn-1}\}$ be an almost orthogonal resolution for G'. Add a new element y_i to each k-tuple in row i of G', $1 \le i \le n$, and call the resulting array G. We label the first $kn - 2$ columns of G, $A_1, A_2, \ldots, A_{kn-2}$, and the last column of G we label X. (X_i denotes the block in row i of X.) The almost orthogonal resolution D' provides an almost orthogonal resolution D in G. Let $D = \{D_1, D_2, \ldots, D_{kn-2}, X\}$.

Let K be a $KS_{kn}(kn + 1; 1, 1)$ defined on $V \cup \{\infty\}$ such that the main diagonal of K contains the $(k + 1)$-tuple $C_i \cup \{\infty\}$ in cell (i, i) for $i = 1, 2, \ldots, n$. Replace the $(k+1)$-tuple $C_i \cup \{\infty\}$ with the $(k+1)$-tuple $C_i \cup \{y_i\}$ for $i = 1, 2, \ldots, n$. Let $Y_i = C_i \cup \{y_i\}$ and let $Y = \{Y_1, Y_2, \ldots, Y_n\}$. Let F be a complementary $(1, k; k+1, n, 1)$-frame defined on W where row i and column i of F do not contain the element y_i, $1 \le i \le n$. Place F in the empty cells of K and delete the main diagonal of this new array of $(k + 1)$-tuples. Call the resulting array E. Label the rows of E by $T_1, T_2, \ldots, T_n$ and the columns of E by $S_1, S_2, \ldots, S_n$.

It is straightforward to verify that the collection of $(k+1)$-tuples in $G \cup E \cup Y$ forms a $(kn + n, k + 1, k)$-$BIBD$. We list a pair of resolutions R^1 and R^2 for the design.

$$R^1 = \{X, A_1, A_2, \ldots, A_{kn-2}, T_1 \cup Y_1, T_2 \cup Y_2, \ldots, T_n \cup Y_n\},$$

$$R^2 = \{Y, D_1, D_2, \ldots, D_{kn-2}, S_1 \cup X_1, S_2 \cup X_2, \ldots, S_n \cup X_n\}. \quad \square$$

We have a similar result which uses $NGBTDs$.

Theorem 5.5. *If there is a $NGBTD(n, k)$ with a pair of orthogonal resolutions*

and a $KS_{k+1}(kn + 1; 1, 1)$ *with a complementary* $(1, k; k + 1, n, 1)$-*frame* (*or a complementary* $DNR(n, k + 1, k)$-*BIBD*), *then there is a* $DNR(nk + n + 1, k + 1, k)$-*BIBD*.

Proof. Let $V_1 = \{x_1, x_2, \ldots, x_{nk+1}\}$ and let $V_2 = \{y_1, y_2, \ldots, y_n\}$. Let G' be a $NGBTD(n, k)$ defined on V_1. Suppose column i of G' does not contain the element x_i for $i = 1, 2, \ldots, kn + 1$. To each k-tuple in row i of G' add the element y_i, $1 \le i \le n$. Let G denote the resulting array of blocks of size $k + 1$. The columns of G will be denoted by $G_1, G_2, \ldots, G_{kn+1}$. G_i contains every element of $(V_1 - \{x_i\}) \cup V_2$ precisely once. The orthogonal resolution for G' will partition the blocks of G into $kn + 1$ (near) resolution classes; call these $D_1, D_2, \ldots, D_{kn+1}$. Suppose D_i contains every element of $(V_1 - \{x_i\}) \cup V_2$ precisely once.

Let K' be a $KS_{k+1}(kn + 1; 1, 1)$ defined on V_1. Place a complementary $(1, k; k + 1, n, 1)$-frame (or a complementary $DNR(n, k+1, k)$-*BIBD*) defined on V_2 in the empty cells of K'. Let K denote the resulting array of $(k + 1)$-tuples. Let $T_1, T_2, \ldots, T_n$ denote the rows of K and let $S_1, S_2, \ldots, S_n$ denote the columns of K. Suppose T_i contains every element of $V_1 \cup (V_2 - \{y_i\})$ precisely once and suppose S_i contains every element of $V_1 \cup (V_2 - \{y_i'\})$ precisely once $(i = 1, 2, \ldots, n)$. (If we have used a complementary $(1, k; k + 1, n, 1)$-frame, then $y_i = y_i'$.)

The collection of blocks in $K \cup G$ forms a $(kn + n + 1, k + 1, k)$-*BIBD*. One (near) resolution of this design consists of $\{G_1, G_2, \ldots, G_{kn+1}, T_1, T_2, \ldots, T_n\}$. An orthogonal resolution is given by

$$\{D_1, D_2, \ldots, D_{kn}, S_1, S_2, \ldots, S_n\}. \quad \square$$

Existence results using the constructions in Theorems 5.4 and 5.5 for block size $k = 2$ appear in [9]. Thus far, we have not constructed any examples for $k \ge 3$. Very little is known about the Kirkman squares, $KS_{k+1}(kn + 1; 1, 1)$, or the $(1, k; k + 1, n, 1)$-frames which are required for $k \ge 3$.

For example, the smallest $KS_4(v; 1, 1)$ known to exist is $v = 64$ [5]. In order to apply our constructions, we would need to find a complementary $(1, 3; 4, 21, 1)$-frame.

6. SUMMARY

We have introduced generalized balanced tournament designs and shown that these designs are related to other types of combinatorial designs. We have also described how to use *GBTDs* to construct resolvable, near resolvable and doubly resolvable *BIBDs*. These constructions together with our existence results for *GBTDs* have already produced several new classes of designs. These are described in [10, 11, 12].

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